

Degradability of Fermionic Gaussian Channels

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We study the degradability of fermionic Gaussian channels. Fermionic quantum channels are a central building block of quantum information processing with fermions and the family of Gaussian channels, in particular, is relevant in the emerging field of electron quantum optics and its applications for quantum information. Degradable channels are of particular interest since they have additive quantum capacity. We derive a simple standard form for fermionic Gaussian channels. This allows us fully characterize all degradable n -mode fermionic Gaussian channels. Consequences for the quantum capacity of those channels are discussed.

The transmission of quantum states in space and time is a fundamental physical process. In general, it leads to an irreversible evolution due to the interaction with an uncontrolled environment and is mathematically described by quantum channels [1]. A central objective of quantum information theory is therefore to study the properties of quantum channels and in particular to compute their capacity to transmit classical or quantum information [2]. Channel capacities are difficult to compute since, in general, they require an optimization over entangled inputs to many channels in parallel [3, 4] and are only known for a few channels.

A class of channels for which the complication of calculating quantum capacity does not arise are the *degradable* channels [5], identified in [6]. They are characterized by the property that so little information leaks out to the environment that it can be reproduced even from the noisy channel output. Recently, the notion of degradability has been generalized to weak [7] and approximate [8] degradability, maintaining some of its useful properties.

While most work on quantum channels has considered channels acting on distinguishable particles (qubits or qudits) or bosonic systems, the most natural information carrier in solid state systems are electrons, i.e., fermionic particles, whose anti-commutation and superselection rules modify concepts of quantum information theory such as entanglement [9–13]. Recent impressive experimental advances demonstrated that electrons can be cleanly and individually transported over sizable distances in well-controlled semiconductor systems [14–18], providing an experimental realization of fermionic quantum channels. This motivates our investigation of *fermionic quantum channels*.

Fermionic quantum channels have mostly been studied for non-interacting fermions, leading to the notion of “quasifree” [19, 20] or “Gaussian” channels or “fermionic linear optics” [21–25]. These all emphasize the analogy with the case of Gaussian bosons which are a very fruitful model for quantum optical quantum information processing [26]. Here we study fermionic Gaussian channels

(FGC). We derive a simple standard form which significantly simplifies further analysis. We give a full characterization of all degradable $n \rightarrow n$ mode FGCs and show that there is essentially only one family of such channels, the “lossy channel”, the quantum capacity of which is known.

Let us consider non-interacting fermions with n -dimensional one-particle Hilbert space \mathcal{H} (“ n modes”), described by $2n$ anticommuting Hermitian Majorana operators $c_k, k = 1, \dots, 2n$, satisfying $\{c_k, c_l\} = 2\delta_{kl}$ and associated annihilation $a_k = (c_{2k-1} - ic_{2k})/2$ and creation operators a_k^\dagger .

Fermionic Gaussian states (FGS) are those states for which Wick’s theorem holds [27] (all cumulants are zero). They are fully described by the $2n \times 2n$ *covariance matrix* (CM) defined as

$$\gamma_{kl} = \frac{i}{2} \text{tr}(\rho[c_k, c_l]). \quad (1)$$

The matrix γ is real and antisymmetric. We frequently use the fact that any such matrix can be brought to the simple form $\Lambda = \bigoplus_{j=1}^n \lambda_j J$, with $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ by a special orthogonal transformation: there exists $\lambda_j \in [-1, 1]$ and $O \in \text{SO}(2n)$ such that $\gamma = O\Lambda O^T$. Λ describes n modes in Gibbs states for the Hamiltonian $a_j^\dagger a_j$. In particular, the Gaussian state is pure iff all $\lambda_j = \pm 1$, or, equivalently, iff $\gamma^2 = -\mathbb{1}$ holds.

If we consider a bipartite system of $n + m$ modes, then simplification of γ under *local* operations $O \in \text{SO}(2n) \oplus \text{SO}(2m)$ is of interest. In particular, any pure state Γ can be brought into the Schmidt-form [10] with CM

$$\begin{pmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{21} & \Gamma_{22} \end{pmatrix} \equiv \begin{pmatrix} J_{2l} \oplus \Lambda & (0_{2l \times 2m} \ K)^T \\ (0_{2l \times 2m} \ -K) & \Lambda \end{pmatrix} \quad (2)$$

by such local unitaries. Here $n = l + m$ and we used $J_{2s} = \bigoplus_{j=1}^s J$, $\Lambda = \bigoplus_{j=1}^m \lambda_j J$, and $K = \bigoplus_{j=1}^m \kappa_j \sigma_x$, with $\lambda_j^2 + \kappa_j^2 = 1$. The parameters λ_j specify the amount of

entanglement between the two parties. In particular, if $\lambda_j = 0$ then the mode j is maximally entangled, while if $\lambda_j = \pm 1$ then it is pure and unentangled.

Now let us turn to *Fermionic Gaussian channels*. We consider quantum channels (trace-preserving completely positive maps) that act on a finite set of n fermionic modes and map Gaussian states to Gaussian states. As discussed in [22] they are fully characterized by how they transform the $2n \times 2n$ covariance matrix γ of the input state. An $n \rightarrow m$ mode FGC \mathcal{T} is defined by a $2m \times 2n$ matrix A and an antisymmetric $2m \times 2m$ matrix B as

$$\mathcal{T} \equiv \mathcal{T}_{(A;B)} : \gamma \mapsto A\gamma A^T + B. \quad (3)$$

Equivalently, a channel $\mathcal{T}_{(A;B)}$ can be characterized via its Choi-Jamiołkowski (CJ) state [5], which is given by the state obtained if the channel acts on the first half of a maximally entangled state. For Gaussian channels, the CJ state is Gaussian with CM $M_{(A;B)}$, since the maximally entangled state of $2n$ fermionic modes can be chosen Gaussian ($l = m$ and $\lambda_j = 0, \kappa_j = 1$ in Eq. (2)). This yields a practical necessary and sufficient criterion for $(A; B)$ to define a valid quantum channel [22]: $\mathcal{T}_{(A;B)}$ describes a valid FGC iff the corresponding CJ-CM is a valid CM, i.e., iff $\mathbb{1} + iM_{(A;B)} \geq 0$, which is readily seen (see [28], Lemma S1) to be the case iff

$$\mathbb{1} - iB - AA^T \geq 0. \quad (4)$$

This implies that B is a valid CM and that $\mathbb{1} - AA^T \geq 0$ for FGCs and that $\ker B \supseteq \ker(\mathbb{1} - AA^T)$: the singular values of A must be ≤ 1 and B must vanish on the unit eigenspace of AA^T (the perfectly transmitted modes). This ensures (as will be used later) that $B' = (\mathbb{1} - AA^T)^{-1/2} B (\mathbb{1} - AA^T)^{-1/2}$ is well-defined and a CM (the inverse denotes the Moore-Penrose pseudo-inverse [29] if $\mathbb{1} - AA^T$ has a kernel).

Every quantum channel \mathcal{T} can be represented as the unitary interaction with an environment (prepared in some state ρ_E). Thus \mathcal{T} comes with a second channel,

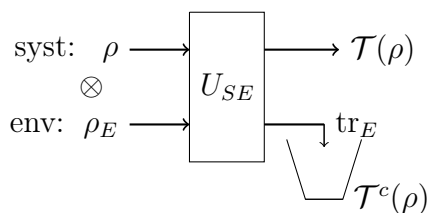


FIG. 1. Channel and complementary channel. For *pure* environmental state ρ_E the dilation U_E and the complementary channel are unique (up to isometries). For mixed ρ_E the channel \mathcal{T}^c is called *weak-complementary* and is not unique.

which describes what information about the input state is leaked into the environment. For pure ρ_E , this channel is unique up to isometries and is called the *complementary channel*, denoted by \mathcal{T}^c .

The relation between \mathcal{T} and \mathcal{T}^c has important consequences for certain capacities of \mathcal{T} [5, 6, 30]. E.g., if it were possible to obtain, for any input ρ , the channel output $\mathcal{T}(\rho)$ also by suitably postprocessing the output of the complementary channel [i.e., if there exists a completely positive map \mathcal{P} such that $\mathcal{P}(\mathcal{T}^c(\rho)) = \mathcal{T}(\rho)$]; such channels are called *antidegradable* [31]) then the channel \mathcal{T} has vanishing quantum capacity, since any finite capacity would lead to a contradiction with the no cloning principle [5].

A more subtle consequence holds if there exists a quantum channel \mathcal{W} such that the concatenation of \mathcal{W} and \mathcal{T} is equal to the complementary channel \mathcal{T}^c . Such channels are called *degradable* and have additive quantum capacity [6]. Degradable and antidegradable channels are at present the only ones for which a good understanding of their quantum capacity can be claimed. As we shall see, the simple structure of fermionic Gaussian channels allows a straight forward answer to the question which $n \rightarrow n$ channels are degradable. Our main result is summarized in the following theorem:

Theorem 1 (Degradable $n \rightarrow n$ Fermionic Gaussian Channels). *All $n \rightarrow n$ degradable FGCs are, up to unitary pre- and post-processing, of the form*

$$\mathcal{T}_p : \gamma \mapsto p\gamma + (1-p)J, \quad (5)$$

where $p \geq \frac{1}{2}$ or a direct sum of such channels.

We show this in three steps: first we observe that concatenating a channel with unitary channels (“unitary pre- and postprocessing”) does not affect the degradability properties. This allows to greatly simplify the further discussion by focusing on FGCs in *standard form* in which the matrix A is diagonal with descendingly ordered positive eigenvalues. This form can always be reached by concatenating the channel with Gaussian unitaries that effect the singular value decomposition (SVD) of $A = O_1 D O_2$ (see [28], Lemma S3). Likewise, it is easy to see that FGCs that act independently of two subsets of modes, i.e., $\mathcal{T}_{(A;B)}$ with $A = A_1 \oplus A_2$ and $B = B_1 \oplus B_2$, are (anti)degradable iff both $\mathcal{T}_{(A_i;B_i)}$ are (see [28], Lemma S4), which, without loss of generality, allows to consider only channels where A has at most one singular value 1. We then show that it is necessary for degradability that (i) $D \geq 1/\sqrt{2}$ and (ii) the channel has a small Choi rank [5], more specifically, its minimal dilation requires no more than n modes. Finally, we prove that such channels cannot be degradable unless $D \propto \mathbb{1}$ or the environmental state CM γ_p decomposes into $\oplus_k \gamma_{p,k}$ and $D = \oplus_k (\alpha_k \mathbb{1})$.

Without loss of generality we consider an $n \rightarrow m$ FGC $\mathcal{T}_{(A;B)}$ with

$$A = \begin{pmatrix} D & 0_{2(n-m)} \end{pmatrix} \quad \text{or} \quad A = \begin{pmatrix} D \\ 0_{2(m-n)} \end{pmatrix} \quad (6)$$

depending on whether $m \leq n$ or $m \geq n$. Here $1 \geq D \geq 0$ is a square matrix with dimension $2\min\{n, m\}$.

If D has $L \geq 2$ eigenvalues equal to 1 it implies that $l = \lfloor L/2 \rfloor$ modes are transmitted perfectly. As B then vanishes on all those modes, the channel is a particular case of a Gaussian product channel $(A; B) = (\mathbb{1} \oplus A_2; 0 \oplus B_2)$ and thus it is degradable iff $\mathcal{T}_{(A_2; B_2)}$ is, where A_2 now has at most one singular value equal to 1. Thus it suffices to consider channels with at most one singular value 1 in the proof of our theorem.

Before we do so, we have to obtain the complementary channel to $\mathcal{T}_{A;B}$ in order to express degradability in terms of A and B . To this end it is useful to find a unitary dilation of $\mathcal{T}_{A;B}$. From Eq. (4) we see that for valid channels $AA^T \leq \mathbb{1}$ and $\ker(\mathbb{1} - AA^T) \subseteq \ker(B)$, and $B' := (\mathbb{1} - AA^T)^{-\frac{1}{2}} B (\mathbb{1} - AA^T)^{-\frac{1}{2}}$ is a valid CM. Then it is easy to check that the $n \rightarrow m$ FGC $\mathcal{T}_{A;B}$ can be obtained by coupling the system with a Gaussian unitary represented by $O_{SE'} \in O(2n + 2m)$,

$$O_{SE'} = \begin{pmatrix} A & \sqrt{\mathbb{1}_{2m} - AA^T} \\ -\sqrt{\mathbb{1}_{2n} - A^T A} & A^T \end{pmatrix}, \quad (7)$$

to an m -mode environment in the Gaussian state with CM B' . To obtain the complementary channel, however, the environment should be *pure*. Let $l \leq n$ denote the number of pure modes of B' , i.e., $B' = O(J_{2l} \oplus L)O^T$ for $0 \leq \lambda_j < 1$, $L = \bigoplus_{j=1}^{m-l} \lambda_j J_2$, and $O \in SO(2m)$. Then

$$\gamma_E = [O \oplus \mathbb{1}_{2(m-l)}] \begin{pmatrix} J_{2l} & & \\ & L & K \\ & -K & L \end{pmatrix} [O^T \oplus \mathbb{1}_{2(m-l)}], \quad (8)$$

where $K = \bigoplus_j \kappa_j \sigma_x$ and $\lambda_j^2 + \kappa_j^2 = 1$ is a purification of B' and $\mathcal{T}_{A;B}$ can be obtained by coupling with

$$O_{SE} = O_{SE'} \oplus \mathbb{1}_{2(m-l)} \quad (9)$$

to the $2m - l$ -mode pure environment in state γ_E .

There are many other physical representations of $\mathcal{T}_{A;B}$ with pure environment γ'_E . However, they are all related isometrically to each other (i.e., by unitary pre- and post-processing and adjoining/removing pure modes) and for our purposes all such representations are equivalent.

With the above representation of $\mathcal{T}_{A;B}$ we can read off the to $\mathcal{T}_{A;B}$ complementary channel. It is given by the $n \rightarrow n + m - l$ channel

$$\mathcal{T}^c : \gamma \mapsto A_c \gamma A_c^T + B_c, \quad (10)$$

where

$$A_c = \begin{pmatrix} \sqrt{\mathbb{1} - A^T A} \\ 0 \end{pmatrix}; \quad B_c = (A^T \oplus \mathbb{1}) \gamma_E (A \oplus \mathbb{1})$$

with γ_E as in Eq. (8).

The question of the degradability of the FGC $\mathcal{T}_{A;B}$ is then, simply, if there exists an $m \rightarrow n + m - l$ FGC $\mathcal{T}_{A_w; B_w}$

such that $\mathcal{T}_{A_w; B_w} \circ \mathcal{T}_{A_c; B_c} = \mathcal{T}_{A; B}$. The degrading map follows directly from $\mathcal{T}_{A; B}$ and $\mathcal{T}_{A_c; B_c}$. It only exists if A has no kernel and is then given by $\gamma \mapsto \tilde{A} \gamma \tilde{A}^T + \tilde{B}$ with

$$\tilde{A} = \begin{pmatrix} \sqrt{\mathbb{1} - A^T A} \\ 0 \end{pmatrix} A^{-1} = \begin{pmatrix} A^{-1} \sqrt{\mathbb{1} - AA^T} \\ 0 \end{pmatrix}, \quad (11)$$

$$\tilde{B} = (A^T \oplus \mathbb{1}) \gamma_E (A \oplus \mathbb{1}) - [(A^{-1} - A^T) \oplus 0] \gamma_E [(A^{-T} - A) \oplus 0]. \quad (12)$$

Using Eq. (4) we see that $\mathcal{T}_{(\tilde{A}; \tilde{B})}$ is completely positive if and only if

$$\tilde{M} \equiv \mathbb{1} - \tilde{A} \tilde{A}^T - i \tilde{B} \geq 0. \quad (13)$$

Thus, quite plausibly, all singular values of A must be $\geq 1/\sqrt{2}$, i.e., losses into the environment must be no more than 50% for degradable channels. In particular, $n \rightarrow m$ channels for $m < n$ are never degradable.

If $m = n$ we can give a full characterization of degradable FGCs. First, we claim that *a $n \rightarrow n$ FGC is not degradable unless its Choi rank is no larger than n modes*. Assuming standard form $A = D$, re-expressing \tilde{M} in (13) in terms of D and the pure $2n - l$ mode environmental state γ_E of a minimal dilation we obtain after repeated application of Schur's lemma (see Supplementary Material [28], Lemma S1) the inequality

$$2D^{-2} - D^{-4} - [O(0_{2l} \oplus \mathbb{1}_{2(n-l)})O^T] \geq 0, \quad (14)$$

as a necessary condition for degradability, where $O \in SO(2n)$ depends on γ_E (for the details see the Supplementary Material [28]). This inequality cannot be fulfilled unless $l = n$, i.e. the environment is no larger than the system. To see this, we use a simple condition on the eigenvalues of two Hermitian matrices and their sum implied by Horn's conjecture, [32, 33]: Let λ_i, μ_j, ν_k denote the descendingly ordered eigenvalues of the Hermitian matrices $X, Y, X + Y$, respectively. Then we have [33]

$$\nu_k \leq \lambda_i + \mu_j \quad \forall i + j = k + 1. \quad (15)$$

For our purposes we take $X = 2D^{-2} - D^{-4}$ and $Y = -Q_1(0_{2m} \oplus \mathbb{1})Q_1^T$ and pick $j = 2l + 1$. Then $\mu_j = -1$ and for all $i > 1$ we have

$$\nu_{2l+i} \leq \lambda_i + \mu_{2l+1} = \frac{2}{d_i^2} - \frac{1}{d_i^4} - 1 \quad \forall i = 1, \dots, 2(n-l).$$

Unless $n - l = 0$ (pure environment) we can take $i = 2$ which means that $d_i < 1$ in which case the expression on the RHS is negative.

Let us now focus on $n \rightarrow n$ FGC with a n -mode environment. To complete the proof of Theorem 1, we show

Lemma 1 (Only constant-loss channels are degradable). *A $n \rightarrow n$ FGC $\mathcal{T}_{(D; B)}$ in standard form with fermionic Gaussian Choi rank n is degradable if and only if $D = \bigoplus_j (d_j \mathbb{1}_{2n_j})$ (i.e., all eigenvalues of D have an even degeneracy), $d_j \geq 1/\sqrt{2}$, and $B = \bigoplus_j B_j$ (B does not contain correlations between blocks pertaining to nondegenerate values $d_j \neq d_k \implies B_{jk} = 0$).*

PROOF: Following analogous arguments to the construction above (in particular, using that $B = \sqrt{1 - D^{-2}}\gamma_p\sqrt{1 - D^{-2}}$ for a n -mode pure-state CM γ_p) we find that the degradability condition (13) becomes

$$2\mathbb{1} - D^{-2} - i \left[D\gamma_p D - \left(\frac{1}{D} - D \right) \gamma_p \left(\frac{1}{D} - D \right) \right] \geq 0. \quad (16)$$

We show now that this is only the case if $D \geq 1/\sqrt{2}$ and if $(\gamma_p)_{kl} = 0$ whenever $d_k \neq d_l$, i.e., if γ_p is a direct sum of independent pure states $\oplus \gamma_{p,k}$ and D a corresponding direct sum of terms proportional to $\mathbb{1}$.

We already saw that $D \geq 1/\sqrt{2}$ is necessary for degradability. If there are one or more eigenvalues $d_i = 1/\sqrt{2}$, then the real part of (16) has a kernel and we see immediately that the inequality can only hold if $\gamma_{ij} = \gamma_{ji} = 0$ for all j such that $d_j \neq 1/\sqrt{2}$. I.e., a channel with some $d_j = 1/\sqrt{2}$ can only be degradable if $D = D' \oplus \frac{1}{\sqrt{2}}\mathbb{1}$ and $\gamma = \gamma_1 \oplus \gamma_2$ in accordance with Theorem 1 (by purity and antisymmetry, both blocks have to have even dimension).

We now assume $D > 1/\sqrt{2}$; multiplying Eq. (16) by $\frac{1}{\sqrt{2-D^{-2}}}$ from left and right, the imaginary part becomes $\gamma_p + R$, where

$$R = -\gamma_p + \frac{1}{W} [\gamma_p - D^2\gamma_p - \gamma_p D^2] \frac{1}{W} \quad (17)$$

with $W = \sqrt{2D^2 - 1}$.

Since γ_p is pure, $i\gamma_p$ has spectrum $\{\pm 1\}$ with eigenprojectors P_{\pm} . Thus Ineq. (16) becomes

$$2P_+ + iR \geq 0, \quad (18)$$

which shows that the overlap $\text{tr}(P_- R) = -i\text{tr}(\gamma_p R)$ must vanish. As detailed in [28], the matrix R is the pointwise (Hadamard-) product with a symmetric matrix r : $R_{kl} = r_{kl}\gamma_{kl}$, and the r_{kl} are strictly negative whenever $d_k \neq d_l$. Simple algebra that makes use of the symmetry of r , antisymmetry and purity of γ then shows (see[28]) that this imposes $\gamma_{kl} = 0$ whenever $d_k \neq d_l$, i.e., the direct-sum decomposition of on D and γ_p into blocks of even dimension corresponding to constant d_k . \square

A final simplification used in Theorem 1 is that the pure state of the environment can without loss of generality be taken to be the vacuum state (with CM $\gamma_E = J$). This is the case since for $A = \sqrt{1-p}\mathbb{1}$ the FGCs $\mathcal{T}_{A;p\gamma_E}$ and $\mathcal{T}_{A;pJ}$ differ only by unitary pre- and postprocessing: for $\gamma_E = OJO^T$ we have $\mathcal{T}_{A;p\gamma_E} = \mathcal{T}_O \circ \mathcal{T}_{A;pJ} \circ \mathcal{T}_{O^T}$.

In summary, we have shown that there is essentially only one family of degradable fermionic Gaussian channels, namely the single-mode attenuation channel $\mathcal{T}_p : \gamma \mapsto (1-p)\gamma + pJ$ (with losses $p \in [0, 1/2]$). Hence FGCs have a much simpler degradability structure than their qubit or even their bosonic Gaussian counterpart [34]. In contrast to the case of qubits [5], there are no degradable n -mode FGCs with large environment. The greater simplicity compared to bosonic Gaussian channels may be a consequence of the simpler normal form

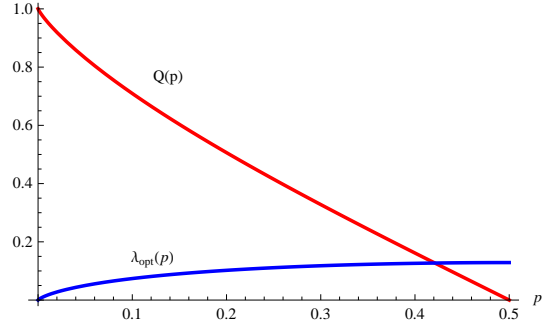


FIG. 2. Quantum capacity of the single-mode lossy channel for loss parameter $p = \alpha^2 \in [1/2, 1]$ (red). The blue line shows the value of λ_{opt} characterizing the optimal input state.

which is attainable by the orthogonal phase space transformations applicable in the fermionic case (rather than what symplectic transformations allow for bosons: [35]).

We can exploit the degradability of \mathcal{T}_p to compute its quantum capacity, $Q(\mathcal{T}_p)$, given by the channels coherent information [2]: $Q(\mathcal{T}_p) = \max_{\gamma} \{S(\mathcal{T}_p(\gamma)) - S((\mathcal{T}_p \oplus \mathbb{1})(\Gamma))\}$, where Γ is a purification of γ . That we can restrict to Gaussian input is a consequence of the extremality of Gaussian states as shown in [36, 37] (for bosons) and generalized to fermions in [38] (see also [28]). Specifically, with $\gamma = \lambda J$ (general one-mode CM) $\mathcal{T}(\gamma)$ has eigenvalues $\pm i(p + (1-p)\lambda)$; we can take $\Gamma = \begin{pmatrix} \lambda J & \sqrt{1-\lambda^2}X \\ -\sqrt{1-\lambda^2}X & \lambda J \end{pmatrix}$ and find that $(\mathcal{T} \oplus \mathbb{1})(\Gamma)$ has one eigenvalue of unit modulus (one pure mode) and the other is $\pm i(1-p+p\lambda)$. The maximum value of the conditional entropy as well as the value λ_{opt} for which it is obtained are shown as a function of the losses p in Fig. 2. This quantum capacity has already been computed in [31], where the qubit channel with Kraus operators $K_1 = |0\rangle\langle 0| + (1-p)|1\rangle\langle 1|$ and $K_2 = \sqrt{p}|0\rangle\langle 1|$ was considered to which \mathcal{T}_p is equivalent.

There are several interesting directions for further research: The generalization of the above result to the case of $n \rightarrow m$ channels is important, in particular even for the *antidegradability* of $n \rightarrow n$ channels, since, in general, the complementary channel is a map between systems of different number of modes (e.g., $n \rightarrow m' = n + m - l$ for the dilation given above).

While it is clear that $n \rightarrow m$ channels with $m < n$ are never degradable (since A has a kernel), in the case $m > n$, the positivity of the real part of Eq. (14) is no longer so easy to decide since it may depend on the details of γ_E and examples of degradable channels with Choi rank larger than $\max\{n, m\}$ exist in this case (see [28]). For the case of $n \rightarrow n$ channels with Choi rank $n' \leq n$ our result also clarifies which of these channels are antidegradable, since in this case the complementary channel is a $n \rightarrow n'$ channel with Choi rank n' and The-

orem 1 applies. Thus, such a FGC $\mathcal{T}_{(A;B)}$ is antidegradable, iff (in standard form) $\sqrt{\mathbb{1} - A^T A} \propto \mathbb{1}$ and $\geq 1/\sqrt{2}$ (or a direct sum of such channels), i.e., $A = \oplus_k d_k \mathbb{1}$ with $d_k \leq 1/\sqrt{2}$.

Can we use our results to put bounds on the quantum capacity of some non-Gaussian channels? It may be possible to adapt results of [37] (for bosons) in order to obtain a lower bound on quantum capacity of general fermionic channels, but the possibility of a fermionic Gaussian teleportation protocol and channel remain to be worked out [38, 39].

A different approach could exploit the recently introduced notion of approximate degradability [8], which describes a class of channels that meet the degradability condition modulo finite $\epsilon \geq 0$ and approximately reproduces all the properties of degradable channels. Our results might also help in the analysis of channels that are a classical mixture of Gaussian channels. These can arise, for example, in certain dephasing scenarios [25, 40–42] and may be understood as a concatenation of a non-Gaussian and Gaussian channels and are only degradable if both components are.

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Supplementary material: Degradability of Fermionic Gaussian Channel

In this Supplementary Material we collect some simple statements and details of the proofs used in the main text. After some useful general lemmas on the properties of block matrices we provide technical details for the proof of the degradability theorem. We conclude with stating and proving an extremality theorem for fermionic Gaussian states.

SOME USEFUL LEMMAS ON MATRICES

Lemma S1 (Positivity conditions). (1) *The hermitian matrix*

$$W = \begin{pmatrix} X & Y \\ Y^\dagger & Z \end{pmatrix}$$

we have that $W \geq 0$ if and only if

$$\ker(Z) \subseteq \ker(Y) \quad \text{and} \quad X - YZ^{-1}Y^\dagger \geq 0. \quad (\text{S1})$$

(2) *For the case of real matrices X, Y, Z with $Z = X$ and $Y = -Y^T$ we also have that $W \geq 0$ if and only if $X + iY \geq 0$.*

PROOF: straight forward calculation of $\binom{a}{b}^\dagger W \binom{a}{b} \geq 0 \forall a, b$; see, e.g., [S1]. \square

This is useful for confirming whether an antisymmetric matrix M is a valid CM as follows: $M \equiv M_{(A;B)}$ is a CM iff $\mathbb{1} - MM^T \geq 0$ which is by (1) the case iff $\begin{pmatrix} \mathbb{1} & M \\ -M^T & \mathbb{1} \end{pmatrix} \geq 0$, which, in turn, holds by (2) iff $\mathbb{1} + iM \geq 0$; inserting the block structure of $M_{(A;B)}$ and using again (1) we finally that this holds iff $\mathbb{1} - AA^T + iB \geq 0$.

Lemma S2 (Bipartite Gaussian Unitaries). *Let U be a Gaussian unitary acting on $n + m$ modes and let the orthogonal matrix*

$$O = \begin{pmatrix} O_{11} & O_{12} \\ O_{21} & O_{22} \end{pmatrix}$$

be its phase space representation. Then we can find orthogonal operations $Q_i, R_i, i = 1(2)$ acting on the first (second) set of modes such that (we assume $n \leq m \equiv n + k$)

$$O = (Q_1 \oplus Q_2) \begin{pmatrix} D & \sqrt{\mathbb{1} - D^2} & 0 \\ -\sqrt{\mathbb{1} - D^2} & D & 0 \\ 0 & 0 & \mathbb{1}_{2k} \end{pmatrix} (R_1 \oplus R_2),$$

where $\mathbb{1} \geq D \geq 0$ is diagonal.

PROOF: immediate consequence of the orthogonality conditions and the singular value decomposition [S2] of the real matrices O_{ij} . \square

If $n < m$ this shows that the n modes of the first system interact with no more than n modes of the second system (those in the range of O_{12}) while the modes in the kernel of O_{12} are “spectator modes”. Their only influence on the state in the first system after application of O stems from their prior entanglement with the “interacting” modes.

QUANTUM CHANNELS

Lemma S3 (Degradability unaffected by concatenation with unitaries). *The quantum channel \mathcal{T} is (anti)degradable iff for any two unitary channels $\mathcal{U}_1, \mathcal{U}_2$ the channel $\mathcal{T}' = \mathcal{U}_2 \circ \mathcal{T} \circ \mathcal{U}_1$ is (anti)degradable.*

PROOF: If \mathcal{D} degrades \mathcal{T} to its complementary channel \mathcal{T}_c then $\mathcal{D}' = \mathcal{D} \circ \mathcal{U}_2^{-1}$ degrades \mathcal{T}' to its complementary channel $\mathcal{T}'_c = \mathcal{T}_c \circ \mathcal{U}_1$. Since \mathcal{U}_i are invertible the converse also holds. \square

Lemma S4 (Degradability of Composite Channels). *Two quantum channels \mathcal{R}, \mathcal{S} are both degradable if and only if the combined channel $\mathcal{R} \otimes \mathcal{S}$ is degradable.*

PROOF: If $\tilde{\mathcal{R}}, \tilde{\mathcal{S}}$ are degrading CPMs for \mathcal{R}, \mathcal{S} , resp., then $\tilde{\mathcal{T}} = \tilde{\mathcal{R}} \otimes \tilde{\mathcal{S}}$ is CP and the degrading map for $\mathcal{T} = \mathcal{R} \otimes \mathcal{S}$. Conversely, if $\tilde{\mathcal{T}}$ is a CPM degrading $\mathcal{T} = \mathcal{R} \otimes \mathcal{S}$ then the CPMs $\tilde{\mathcal{R}} = \mathcal{R}_2 \circ \tilde{\mathcal{T}} \circ \mathcal{R}_1$ (with $\mathcal{R}_1 : \rho \mapsto \rho \otimes \mathcal{S}(\Phi)$ and $\mathcal{R}_2 : \rho_{1'2'} \mapsto \text{tr}_{2'}(\rho_{1'2'})$ for an arbitrary state Φ) and $\tilde{\mathcal{S}}$ (defined similarly) satisfy $\tilde{\mathcal{R}} \circ \mathcal{R}(\rho) = \text{tr}_{2'}(\tilde{\mathcal{T}}(\mathcal{R}(\rho) \otimes \mathcal{S}(\Phi))) = \text{tr}_{2'}(\tilde{\mathcal{T}}(\mathcal{T}(\rho \otimes \Phi))) = \mathcal{R}_c(\rho)$ (and analogously for $\tilde{\mathcal{S}}$), i.e., they degrade \mathcal{R} resp. \mathcal{S} . \square

Lemma S5 (Perfectly transmitted Majorana modes). *A FGC with $k \geq 2l$ perfectly transmitted Majorana modes is unitarily equivalent to a FGC of the form $T = T_{(\mathbb{1}_{2l}; 0)} \oplus T_{D', B'}$.*

This will be useful for the degradability classification later, since by Lemma S4 it then suffices to study the smaller channel $T_{D', B'}$.

An interesting characteristic of the channel is the *smallest* environment such that a physical representation with a pure state is possible. The dimension of that environment is called the *Choi rank* of the channel [S3] and is given by the rank of the Choi-Jamiołkowski

state of the channel. For our fermionic case we consider the state obtained by letting the channel $T_{A;B}$ act on the first n modes of a $2n$ -mode maximally entangled fermionic Gaussian (e.g., with CM as in Eq. (2) with $l = 0$ and $\lambda_j = 1$). In the following, it is convenient to measure the Choi rank via the *minimum number of fermionic modes* needed to purify this Gaussian state. For an $n \rightarrow n$ FGC $T_{A;B}$ one can check that its Choi rank is $2n - \dim \ker(\mathbb{1} - AA^T - iB)$:

Lemma S6 (Choi rank of $n \rightarrow n$ Gaussian channels). *For an $n \rightarrow n$ channel $T_{(A;B)}$ in standard form, its Choi rank is given by half the number of eigenvalues of $M_{(A;B)}$ with modulus smaller than 1, that is, by $2n - l$ modes where $l = \dim \ker(\mathbb{1} - iM_{(A;B)}) = \dim \ker(\mathbb{1} - AA^T - iB)$. If $\mathbb{1} - AA^T$ has full rank then $l = \ker(\mathbb{1} - iB)$.*

PROOF: The CJ-state of the channel $T_{A;B}$ is the $2n$ -mode Gaussian state $\text{rank} \rho_{M_{A;B}}$. Since $M = O(\oplus_k \lambda_k J)O^T$, $\text{rank} \rho_{M_{A;B}}$ is unitarily equivalent to $\rho_{\oplus_k \lambda_k J}$, and thus $\text{rank} \rho_{M_{A;B}} = \mathcal{F}_-(\mathbb{C}^{2n-l})$, where l is the number of eigenvalues $|\lambda_k| = 1$, i.e., the number of pure modes in $\rho_{M_{A;B}}$. In terms of its CM we have $l = \dim(\ker(\mathbb{1} - iM_{(A;B)}))$ (obvious for $O = \mathbb{1}$). Using the block structure of $\mathbb{1} - iM$ we see that $(\mathbb{1} - iM_{(A;B)}) \begin{pmatrix} x \\ y \end{pmatrix} = 0$ iff $y = -iA^T x$ and $x \in \ker(\mathbb{1} - iB - AA^T)$. If $\mathbb{1} - AA^T$ has full rank, then $W = (\mathbb{1} - AA^T)^{-1/2}$ is a well-defined similarity transformation and hence the kernels of $(\mathbb{1} - AA^T - iB)$ and $W(\mathbb{1} - AA^T - iB)W^T = (\mathbb{1} - iB')$ have the same dimension l . \square

The following three Lemmas constitute the core of the proof of our main theorem.

Lemma S7 (Small Choi-rank necessary for degradability). *A $n \rightarrow n$ FGC is not degradable unless its Choi rank is no larger than n modes.*

PROOF: The positivity of $\tilde{M} = \mathbb{1} - \tilde{A}\tilde{A}^T - i\tilde{B}$ (cf. Eqs. (11-13) of the main text) can be checked using Lemma S1. To apply it, we view \tilde{M} as a 2×2 block matrix with blocks $\tilde{X}, \tilde{Y}, \tilde{Z}$ and also write the CM of the pure environmental state in that form: $\gamma_E = \begin{pmatrix} E_{11} & E_{12} \\ -E_{12}^T & E_{22} \end{pmatrix}$. Then we have

$$\begin{aligned} \tilde{X} &= 2\mathbb{1} - D_A^{-2} - iD_A E_{11} D_A + \\ &\quad + i(D_A^{-1} + D_A)E_{11}(D_A^{-1} + D_A), \\ \tilde{Y} &= -iD_A E_{12}, \\ \tilde{Z} &= \mathbb{1} - iE_{22}. \end{aligned}$$

Now we consider a minimal unitary dilation of $T_{A;B}$ on $2n-l$ environmental modes. We use for the pure $n \times (n-l)$ bipartite FGS γ_E the standard form as given in Eq. (2)

and write

$$\begin{aligned} E_{11} &= Q_1(J_{2l} \oplus L)Q_1^T, \\ E_{22} &= Q_2 L Q_2^T, \\ E_{12} &= Q_1 \begin{pmatrix} 0_{2l \times 2(n-l)} \\ K \end{pmatrix} Q_2^T, \end{aligned}$$

with $Q_1 \in \text{SO}(2n)$, $Q_2 \in \text{SO}(2(n-l))$ and $L = \oplus_{j=1}^{n-l} \lambda_j J$, $K = \oplus_j \kappa_j^{n-l} X$ with $\lambda_j^2 + \kappa_j^2 = 1$. The blocks refer to the $l \leq n$ pure and $n-l$ mixed modes in B' (i.e., those environmental modes coupled to directly by the system) and the $n-l$ environmental spectator modes needed to purify B' . Hence we have $|\lambda_j| < 1, |\kappa_j| > 0$. Therefore, $\tilde{Z} = \mathbb{1} - iE_{22} = Q_2(\mathbb{1} - iL)Q_2^T$ has no kernel, i.e., we can use the Schur complement formula Eq. (S1) to check positivity of \tilde{M} . The term $\tilde{Y}\tilde{Z}^{-1}\tilde{Y}^\dagger$ simplifies to $D_A Q_1 \begin{pmatrix} 0_{2l} \\ K(\mathbb{1} - iL)^{-1} K^T \end{pmatrix} Q_1^T D_A$ and using the block-diagonal form of K, L we find $K(\mathbb{1} - iL)^{-1} K^T = \mathbb{1} - iL$, giving the condition

$$2\mathbb{1} - D_A^{-2} - D_A Q_1 \begin{pmatrix} 0 \\ \mathbb{1}_{2(n-l)} \end{pmatrix} Q_1^T D_A + i(\dots) \geq 0, \quad (\text{S2})$$

where the imaginary part is not relevant in the following. If $l = 0$ (no pure modes), the condition for the positivity of \tilde{M} simplifies to $2\mathbb{1} - D_A^{-2} - D_A^2 + i(\dots) \geq 0$, and the real part has negative eigenvalues (unless $D_A = \mathbb{1}$, a case we excluded by Lemma S4), thus Ineq. (S2) does not hold and the channel is not degradable in that case. To extend this to all $l < n$ we multiply Ineq. (S2) from both sides with D_A^{-1} (we already know $D_A \geq 1/\sqrt{2}$) and the real part of the LHS becomes

$$2D_A^{-2} - D_A^{-4} - Q_1(0_{2l} \oplus \mathbb{1}_{2(n-l)})Q_1^T. \quad (\text{S3})$$

To see that this cannot be positive unless $l = n$ (pure environment) we use a simple condition on the eigenvalues of two hermitian matrices and their sum implied by Horn's conjecture, [S4, S5]: Let λ_i, μ_j, ν_k denote the descendingly ordered eigenvalues of the hermitian matrices $X, Y, X + Y$, respectively. Then we have [S5]

$$\nu_k \leq \lambda_i + \mu_j \quad \forall i + j = k + 1. \quad (\text{S4})$$

For our purposes we take $X = 2D_A^{-2} - D_A^{-4}$ and $Y = -Q_1(0_{2l} \oplus \mathbb{1})Q_1^T$ and pick $j = 2l + 1$ such that $\mu_j = -1$. Then for all $i > 1$ we have

$$\nu_{2l+i} \leq \lambda_i + \mu_{2l+1} = \frac{2}{d_i^2} - \frac{1}{d_i^4} - 1 \quad \forall i = 1, \dots, 2(n-l).$$

Unless $n-l = 0$ (pure environment) we can take $i = 2$ which means that $d_i < 1$ in which case the expression on the RHS is negative. \square

Lemma S8 (Only constant-loss channels are degradable). *A $n \rightarrow n$ FGC $T_{(D;B)}$ in standard form with*

Choi rank n is degradable if and only if $D = \oplus_j (d_j \mathbb{1}_{2n_j})$ (i.e., all eigenvalues d_j of D have an even degeneracy), $d_j \geq 1/\sqrt{2}$, and $B = \oplus_j B_j$ (B does not contain correlations between blocks pertaining to nondegenerate values: $d_j \neq d_k \implies B_{jk} = 0$).

PROOF: We have $A = D$ diagonal and $B = \sqrt{\mathbb{1} - D^{-2}} \gamma_p \sqrt{\mathbb{1} - D^{-2}}$ for a n -mode pure-state CM γ_p ; the degrading map is given by $\tilde{A} = \sqrt{\mathbb{1} - D^{-2}}$ and $\tilde{B} = D \gamma_p D - (D^{-1} - D) \gamma_p (D^{-1} - D)$ and the condition for degradability becomes

$$2\mathbb{1} - D^{-2} + i \left[D \gamma_p D - \left(\frac{1}{D} - D \right) \gamma_p \left(\frac{1}{D} - D \right) \right] \geq 0. \quad (\text{S5})$$

We show now that this is only the case if $D \geq 1/\sqrt{2}$ and if $(\gamma_p)_{kl} = 0$ whenever $d_k \neq d_l$, i.e., γ_p a direct sum of independent pure states $\oplus_{\gamma_{p,k}}$ and D a corresponding direct sum of terms proportional to $\mathbb{1}$.

We already saw that $D \geq 1/\sqrt{2}$ is necessary for degradability. If there is an eigenvalue equal to $1/\sqrt{2}$, say $d_i = 1/\sqrt{2}$, then the real part of Ineq. (S6) has a kernel and we see immediately that the inequality can only hold if $\gamma_{ij} = 0$ for all j such that $d_j \neq 1/\sqrt{2}$. I.e., a channel with some $d_j = 1/\sqrt{2}$ can only be degradable if $D = D' \oplus \frac{1}{\sqrt{2}} \mathbb{1}$ and $\gamma = \gamma_1 \oplus \gamma_2$ in accordance with our theorem.

The condition for degradability for Choi-rank n channel becomes in standard form:

$$2\mathbb{1} - D^{-2} - i \left[D \gamma_p D - \left(\frac{1}{D} - D \right) \gamma_p \left(\frac{1}{D} - D \right) \right] \geq 0, \quad (\text{S6})$$

for $D \geq 1/\sqrt{2}$ diagonal and γ_p the pure state such that $B = \sqrt{\mathbb{1} - D^{-2}} \gamma_p \sqrt{\mathbb{1} - D^{-2}}$.

(a) We can assume $D > 1/\sqrt{2}$ since in case that $d_i = 1/\sqrt{2}$ the real part of Eq. (S6) has a kernel and thus the i th column and row of the imaginary part must vanish. Since all the matrix elements of the imaginary part are proportional to the corresponding element of γ_p , we have $\gamma_{ij} = 0$ for all j with $d_j \neq d_i = 1/\sqrt{2}$.

(b) Then, multiplying Eq. (S6) by $\frac{1}{\sqrt{2-D^{-2}}}$ from left and right and introducing $W = \sqrt{2D^2 - 1}$, we see that it is equivalent to

$$1 + i \frac{1}{W} [\gamma_p - D^2 \gamma_p - \gamma_p D^2] \frac{1}{W} \geq 0. \quad (\text{S7})$$

(c) To see that this necessitates the block structure of γ_p , we write the imaginary part of Eq. (S7) as $\gamma_p + R$ and exploit that γ_p is pure, which implies that $i\gamma_p = P_+ - P_-$ with orthogonal projectors P_{\pm} on the ± 1 -eigenspaces of γ_p that span the full space ($P_+ + P_- = \mathbb{1}$). Thus Eq. (S7) is equivalent to

$$2P_+ + iR \geq 0, \quad (\text{S8})$$

which implies that $\text{tr}(P_- R) = 0$ and since the antisymmetric matrix R is also traceless ($\text{tr}(R) = 0$) we obtain

that it is necessary for Eq. (S6) that

$$\text{tr}(\gamma_p R) = 0. \quad (\text{S9})$$

(d) This condition is sufficient to complete the proof. Let us consider γ, R as blockmatrices with 2×2 blocks denoted by $\gamma^{ij}, R^{ij}, i, j = 1, \dots, n$ and with entries $\gamma_{kl}^{ij}, k, l = 1, 2$ etc. We then use that the elements of R are multiples of the corresponding element of γ_p :

$$R_{kl}^{ij} = \left(-1 + \frac{1 - (d_k^i)^2 - (d_l^j)^2}{\sqrt{2(d_k^i)^2 - 1} \sqrt{2(d_l^j)^2 - 1}} \right) \gamma_{kl}^{ij} \equiv r_{kl}^{ij} \gamma_{kl}^{ij},$$

and it is easily checked [S6] that $r_{kl}^{ij} < 0$ for $d_k^i \neq d_l^j > 1/\sqrt{2}$. The matrices γ and R are antisymmetric, hence $\gamma^{ji} = -(\gamma^{ij})^T$ and $r_{kl}^{ji} = r_{lk}^{ij}$, which allows to simplify the expression for $\text{tr}(\gamma_p R)$:

$$\begin{aligned} \text{tr}(\gamma_p R) &= \sum_{i,j} \text{tr}(\gamma^{ij} R^{ji}) \\ &= \sum_i \text{tr}(\gamma^{ii} R^{ii}) + \sum_i \sum_{j \neq i} \text{tr}(\gamma^{ij} R^{ji}) \\ &= -2 \sum_i r_{12}^{ii} (\gamma_{12}^{ii})^2 - \sum_i \sum_{j \neq i} \sum_{k,l=1}^2 r_{kl}^{ji} (\gamma_{kl}^{ij})^2. \end{aligned}$$

(e) With all $r_{kl}^{ij} \leq 0$ and γ_{kl}^{ij} real, all terms in the above sums are ≥ 0 and the sum is strictly positive unless *all* the terms are zero which requires that $\gamma_{kl}^{ij} = 0$ whenever $d_k^i \neq d_l^j$. Since the d_k^i are monotonically decreasing in standard form, we have $D = \oplus_m d_m \mathbb{1}_{n_m}$ and thus Eq. (S9) implies $\gamma_p = \oplus_m G_m$. The blocks G_m are (of course) antisymmetric and must all have even dimension: odd block are incompatible with γ_p being pure, since the eigenvalues of $i\gamma_p$ are $\{\pm 1\}$ and must coincide with those of the blocks iG_m – but an odd antisymmetric matrix always has at least one eigenvalue zero. Thus all the G_m are valid CMs and we have the advertised block-structure of γ_p and D . \square

Example S1 (A large-Choi-rank $n \rightarrow m$ degradable FGC). If $m > n$ then in standard form $A = \begin{pmatrix} D & 0 \end{pmatrix}^T$ and Eq. (S3) becomes

$$2D^{-2} - D^{-4} - [O(0_{2l} \oplus \mathbb{1}_{2(n-l)}) O^T]_{[2n, 2n]}, \quad (\text{S10})$$

where $[X]_{[2n, 2n]}$ denotes the upper left $2n \times 2n$ block of X . That a Choi rank of $\leq n$ (or $\leq m$) is not necessary for degradability in that case can be seen from a simple example: the $(n \rightarrow n+k)$ -mode FGC $T: \gamma \mapsto (A\gamma A^T + B) \oplus B_2$, where $T_{(A;B)}$ is a $n \rightarrow n$ degradable FGC and B_2 the CM of a full-rank k -mode Gaussian state. We can obviously degrade T by discarding the modes in state B_2 , applying the degrading map (of the channel $T_{A;B}$) to the remaining modes and adjoining a system in a pure state that is a purification of B_2 . On the other hand, the Choi rank of T is the Choi rank of $T_{A;B}$ plus $2k$, i.e., up to $m+k$.

EXTREMALITY OF FERMIONIC GAUSSIAN STATES

We show now that that Gaussian states are *extremal* with respect to certain entanglement measures among all states with the same covariance matrix. This follows in the same way as for bosonic systems [S7] from the fact that FGS can be obtained via a non-commutative central limit and will later be useful to compute the quantum capacity of certain FGCs.

Theorem S1 (Extremality Theorem). *Let $f : \mathcal{B}(\mathcal{H}^{\otimes d}) \rightarrow \mathbb{R}$ be a continuous functional, which is strongly sub-additive and invariant under local unitaries [$f(U^{\otimes d}\rho U^{\dagger\otimes d}) = f(\rho)$]. Then for every even density operator ρ describing d -partite fermionic system we have that $f(\rho) \geq f(\rho_G)$, where ρ_G is the even fermionic Gaussian state with the same second moments as ρ .*

PROOF. Let us start by giving the precise definition of strong super-additivity.

Definition S1. *Let ρ be a density operator on $\mathcal{H} := (\mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2}) \otimes (\mathcal{H}_{B_1} \otimes \mathcal{H}_{B_2})$ and ρ_i , $i = 1, 2$ restrictions of ρ on $\mathcal{H}_{A_i} \otimes \mathcal{H}_{B_i}$. Then the functional $f : \mathcal{B}(\mathcal{H}) \rightarrow \mathbb{R}$ is called super-additive if and only if for all ρ it holds*

$$f(\rho) \geq f(\rho_1) + f(\rho_2) \quad (\text{S11})$$

and equality holds for $\rho = \rho_1 \otimes \rho_2$ (and with a natural generalization to more parties).

Note that we can identify the tensor product of n -mode Fock space and m -mode Fock space with the Fock space of $n + m$ modes: $\mathcal{F}_-(\mathbb{C}^{n+m}) \simeq \mathcal{F}_-(\mathbb{C}^n) \otimes \mathcal{F}_-(\mathbb{C}^m)$.

Analogously we can define so-called strong sub-additivity, which becomes useful when estimating channel capacities.

The basic idea of the proof, as in bosonic case, is the following chain of relations [S7]:

$$f(\rho) = \frac{1}{n} f(\rho^{\otimes n}) = \frac{1}{n} f(\underbrace{U^{\otimes d} \rho^{\otimes n} U^{\dagger\otimes d}}_{\tilde{\rho}}) \quad (\text{S12})$$

$$\geq \frac{1}{n} \sum_{k=1}^n f(\tilde{\rho}_k) \rightarrow f(\rho_G), \quad (\text{S13})$$

where the additivity, invariance under local unitaries and strong super-additivity respectively is used (here $\tilde{\rho}_k$ denotes the reduced state of the modes corresponding to the k th copy). The last step is the result of the quantum mechanical central limit theorem for anti-commuting variables [S8].

Before proving the theorem we need to introduce the notation. We consider n copies of a d -mode fermionic system and denote the associated creation and annihilation operators of the i th mode of the j th copy by $a_i^{j\dagger}$, a_i^j respectively, where $j = 1, \dots, n$, $i = 1, \dots, d$. Here

$a_i^{j\dagger} \equiv (a_i^j)^\dagger$. Note that for simplicity of notation we consider here a d -mode d -partite system, i.e., one mode per site. The extension to $L_j \geq 1$ modes at site j is straight forward.

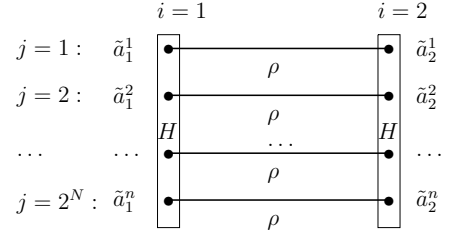


FIG. S1. The local “Gaussification” operation for a bipartite system. Given $n = 2^N$ copies of a bipartite state, they are transformed by a local Gaussian unitary (H), which effects a basis change $a_i^j \rightarrow \tilde{a}_i^j$ such that the bipartite reduced state of the each copy approximates the same bipartite Gaussian state the second moments of which are given by those of the initial (not necessarily Gaussian) state.

The only step requiring a proof in Eq. (S13) is that $\frac{1}{n} \sum_{k=1}^n f(\tilde{\rho}_k)$ converges to $f(\rho_G)$ where ρ_G is the Gaussian state with the same second moments as ρ . For a suitable choice of U , this follows along the lines of the central-limit theorem proved in [S8].

To see this we let $n = 2^N$ and choose U as the local passive Gaussian unitary, which transforms annihilation operators at site i as

$$a_i^j \mapsto \tilde{a}_i^j = \sum_{l=1}^n \frac{H_{jl}}{\sqrt{n}} a_i^l,$$

where $H = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{\otimes N}$. We see that the first mode at site i in the transformed system is now described by the symmetric combinations $\tilde{a}_i^1 = \frac{1}{\sqrt{n}}(a_i^1 + a_i^2 + \dots a_i^n)$ of fermionic operators. In [S8] it was shown that $\tilde{\rho}_1$ converges to ρ_G by showing that all cumulants except those of second order (covariances) vanish and the latter coincide with the covariances of ρ . We need to generalize this to the other, non-symmetric, combinations appearing in modes 2 to n at each site. The operators \tilde{a}_i^j , $j \geq 2$ differ from \tilde{a}_i^1 only by the appearance of minus signs to be applied to all operators referring to one half of the copies involved. But since the different copies describe independent systems (as evidenced by the tensor product structure of the state $\rho^{\otimes n}$) and since for fermionic states only *even* moments of the Fermi operators are non-zero, these minus signs appear only in even powers when computing the cumulants of $\tilde{\rho}_k$, $k \geq 2$ and therefore all the reduced states are the same and by [S8] each of them converges to the same Gaussian state ρ_G as $n \rightarrow \infty$. \square

The detailed convergence argument can be found in [S9].

There are several interesting functionals which satisfy the conditions of the theorem, among them entropy, relative entropy, distillable entanglement [S10], and squashed entanglement [S11].

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[S6] Write $d_k^i = d + x$, $d_l^j = d - x$ and note that $r_{kl}^{ij} =$

$-1 + \frac{1-2(d^2+x^2)}{\sqrt{[1-2(d^2+x^2)]^2-16x^2d^2}}$. The second term is negative since $d^2+x^2 > (d-x)^2 > 1/2$ and the denominator has smaller absolute value than the numerator.

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